AP CALCULUS AB AND BC

UNIT 2

Differentiation: Definition and Fundamental Properties

AP	AP EXAM WEIGHTING	10–12% AB 4–7% BC
	CLASS PERIODS	~13–14 AB ~9–10 BC

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 2

Multiple-choice: ~30 questions
Free-response: 3 questions (partial)

UNIT 2

AP EXAM WEIGHTING	10–12% AB	4–7% BC
CLASS PERIODS	~13–14 AB	~9–10 BC

Differentiation: Definition and Fundamental Properties

BIG IDEA 1
Change CHA
- How can a state determine the rate of change in high school graduates at a particular level of public investment in education (in graduates per dollar) based on a model for the number of graduates as a function of the state's education budget?

BIG IDEA 2
Limits LIM
- Why do mathematical properties and rules for simplifying and evaluating limits apply to differentiation?

BIG IDEA 3
Analysis of Functions FUN
- If you knew that the rate of change in high school graduates at a particular level of public investment in education (in graduates per dollar) was a positive number, what might that tell you about the number of graduates at that level of investment?

Developing Understanding

Derivatives allow us to determine instantaneous rates of change. To develop understanding of how the definition of the derivative applies limits to average rates of change, create opportunities for students to explore average rates of change over increasingly small intervals. Graphing calculator explorations of how various operations affect slopes of tangent lines help students to make sense of basic rules and properties of differentiation. Encourage students to apply the order of operations as they select differentiation rules. Developing differentiation skills will allow students to model realistic instantaneous rates of change in Unit 4 and to analyze graphs in Unit 5.

Building the Mathematical Practices

1.E 2.B 4.C

Mathematicians know that a solution will only be as good as the procedure used to find it and that the difference between being correct and incorrect can often be traced to an arithmetic or procedural error. In other words, mathematicians know that the details matter. Students often find it difficult to apply mathematical procedures—including the rules of differentiation—with precision and accuracy. For example, students may drop important notation, such as a parenthesis, or misapply the product rule by taking the derivative of each factor separately and then multiplying those together. The content of Unit 2 is a foundational entry point for practicing the skill of applying mathematical procedures and learning to self-correct before common mistakes occur.

This is also an opportunity to revisit and reinforce the practice of connecting representations, as students will be seeing derivatives presented in analytical, numerical, graphical, and verbal representations. Students can practice by extracting information about the original function, f, from a graphical representation of f'. This can help prevent misunderstandings when examining the graph of a derivative (such as misinterpreting it as the graph of the original function instead).

Preparing for the AP Exam

Students should practice presenting clear mathematical structures that connect their work with definitions or theorems. For example, when asked to estimate the slope of the line tangent to a curve at a given point based on information provided in a table of values, as in **2013 AP Exam Free-Response Question 3 Part A**, students must present a difference quotient:

$$C'(3.5) \approx \frac{C(4)-C(3)}{4-3} = \frac{12.8-11.2}{1}.$$

Failure to present this structure will cost students the point they might have earned, even with a correct numerical answer. Similarly, to evaluate the derivative of $f(x)=u(x)\cdot v(x)$ at $x=3$, students should show the product rule structure, as in $f'(3)=u(3)v'(3)+v(3)u'(3)$, and import values. Finally, students should present mathematical expressions evaluated on the calculator and use specified rounding procedures, typically rounding or truncating to three places after the decimal point. It is helpful to establish the habit of storing intermediate calculations in the calculator in order to avoid accumulation of rounding errors.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~13–14 CLASS PERIODS (AB) ~9–10 CLASS PERIODS (BC)
CHA-2	**2.1 Defining Average and Instantaneous Rates of Change at a Point**	2.B Identify mathematical information from graphical, symbolic, numerical, and/or verbal representations.	
CHA-2	**2.2 Defining the Derivative of a Function and Using Derivative Notation**	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems. 4.C Use appropriate mathematical symbols and notation (e.g., Represent a derivative using $f'(x)$, y', and $\frac{dy}{dx}$).	
CHA-2	**2.3 Estimating Derivatives of a Function at a Point**	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-2	**2.4 Connecting Differentiability and Continuity: Determining When Derivatives Do and Do Not Exist**	3.E Provide reasons or rationales for solutions and conclusions.	
FUN-3	**2.5 Applying the Power Rule**	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-3	**2.6 Derivative Rules: Constant, Sum, Difference, and Constant Multiple**	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-3 LIM-3	**2.7 Derivatives of $\cos x$, $\sin x$, e^x, and $\ln x$**	1.E Apply appropriate mathematical rules or procedures, with and without technology.	

UNIT AT A GLANCE *(cont'd)*

Enduring Understanding	Topic	Suggested Skills	Class Periods ~13–14 CLASS PERIODS (AB) ~9–10 CLASS PERIODS (BC)
FUN-3	**2.8 The Product Rule**	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	2.9 The Quotient Rule	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	2.10 Finding the Derivatives of Tangent, Cotangent, Secant, and/or Cosecant Functions	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
	Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 2. Review the results in class to identify and address any student misunderstandings.		

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	2.1 2.2 2.3	**Graph and Switch** Present students with two or three functions and the graph of each function. Have each student choose a random derivative question and one function. Questions could include: Find the average rate of change on an interval, instantaneous rate of change at a point, derivative as a function, derivative value at a point, or equations for tangent or normal lines at a point. Have students answer their question and place their answer onto the function's graph. Then have students share their solutions with each other to give and receive feedback.
2	2.4	**Match Mine** Create cards containing graph images of functions with various continuous, discontinuous, differentiable, and nondifferentiable points or intervals. Provide each student in a pair with the same nine cards. Student A arranges their graphs in a 3×3 grid, which is not visible to Student B. Student A describes each of their graph's positions using information about continuity and differentiability to describe the graph. Based on the descriptions, Student B attempts to arrange their cards to match the grid of Student A.
3	2.5 2.6 2.7 2.8 2.9 2.10	**Error Analysis** Assign a function to each student. Ask them to find the function's derivative using one or more derivative rules. Allow them to check their answers. Ask half of the class to redo their work to include an error, thus having the wrong answer. Ask students to record their correct or incorrect work on a card. Mix up the cards and redistribute, having students determine if the answer is correct or incorrect. If incorrect, they should explain what error was made, and find the correct answer.
4	2.5 2.6 2.7 2.8 2.9 2.10	**Graphic Organizer** Provide students with colored paper, pens, and markers. Ask them to create a chart, a foldable card, or other creative method to organize all the derivative rules. For each rule, have them include the mathematical definition, examples, pictures, and helpful hints to understand and remember the rule.
5	2.8 2.9	**Round Table** Provide each student with the same worksheet containing four functions that require the product rule or quotient rule when finding the derivative. Then have students sit in groups of four. Each student determines the derivative of function No. 1, and then they pass their papers clockwise to the next student. Each student checks the first problem and, if necessary, discusses any mistakes with the previous student. Each student now completes function No. 2 on the paper, and the process continues until each student has their original paper back.

TOPIC 2.1
Defining Average and Instantaneous Rates of Change at a Point

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, symbolic, numerical, and/or verbal representations.

Required Course Content

ENDURING UNDERSTANDING

CHA-2

Derivatives allow us to determine rates of change at an instant by applying limits to knowledge about rates of change over intervals.

LEARNING OBJECTIVE

CHA-2.A

Determine average rates of change using difference quotients.

CHA-2.B

Represent the derivative of a function as the limit of a difference quotient.

ESSENTIAL KNOWLEDGE

CHA-2.A.1

The difference quotients $\frac{f(a+h)-f(a)}{h}$ and $\frac{f(x)-f(a)}{x-a}$ express the average rate of change of a function over an interval.

CHA-2.B.1

The instantaneous rate of change of a function at $x=a$ can be expressed by $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ or $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, provided the limit exists. These are equivalent forms of the definition of the derivative and are denoted $f'(a)$.

SUGGESTED SKILLS

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

Communication and Notation

4.C

Use appropriate mathematical symbols and notation.

AVAILABLE RESOURCES

- Professional Development > **Definite Integrals: Interpreting Notational Expressions**
- AP Online Teacher Community Discussion > **How to "Say" Some of the Notation**

TOPIC 2.2

Defining the Derivative of a Function and Using Derivative Notation

Required Course Content

ENDURING UNDERSTANDING

CHA-2

Derivatives allow us to determine rates of change at an instant by applying limits to knowledge about rates of change over intervals.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
CHA-2.B Represent the derivative of a function as the limit of a difference quotient.	**CHA-2.B.2** The derivative of f is the function whose value at x is $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, provided this limit exists. **CHA-2.B.3** For $y=f(x)$, notations for the derivative include $\frac{dy}{dx}$, $f'(x)$, and y'. **CHA-2.B.4** The derivative can be represented graphically, numerically, analytically, and verbally.
CHA-2.C Determine the equation of a line tangent to a curve at a given point.	**CHA-2.C.1** The derivative of a function at a point is the slope of the line tangent to a graph of the function at that point.

TOPIC 2.3
Estimating Derivatives of a Function at a Point

Required Course Content

ENDURING UNDERSTANDING

CHA-2

Derivatives allow us to determine rates of change at an instant by applying limits to knowledge about rates of change over intervals.

LEARNING OBJECTIVE

CHA-2.D

Estimate derivatives.

ESSENTIAL KNOWLEDGE

CHA-2.D.1

The derivative at a point can be estimated from information given in tables or graphs.

CHA-2.D.2

Technology can be used to calculate or estimate the value of a derivative of a function at a point.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Classroom Resource > **Approximation**
- Classroom Resource > **Reasoning from Tabular Data**

SUGGESTED SKILL

Justification

3.E

Provide reasons or rationales for solutions and conclusions.

ILLUSTRATIVE EXAMPLES

For FUN-2.A.2:

- The left hand and right hand limits of the difference quotient are not equal, as in $f(x)=|x|$ at $x=0$.
- The tangent line is vertical and has no slope, as in $f(x)=\sqrt[3]{x}$ at $x=0$.

TOPIC 2.4
Connecting Differentiability and Continuity: Determining When Derivatives Do and Do Not Exist

Required Course Content

ENDURING UNDERSTANDING

FUN-2

Recognizing that a function's derivative may also be a function allows us to develop knowledge about the related behaviors of both.

LEARNING OBJECTIVE

FUN-2.A

Explain the relationship between differentiability and continuity.

ESSENTIAL KNOWLEDGE

FUN-2.A.1

If a function is differentiable at a point, then it is continuous at that point. In particular, if a point is not in the domain of f, then it is not in the domain of f'.

FUN-2.A.2

A continuous function may fail to be differentiable at a point in its domain.

TOPIC 2.5
Applying the Power Rule

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.A

Calculate derivatives of familiar functions.

ESSENTIAL KNOWLEDGE

FUN-3.A.1

Direct application of the definition of the derivative and specific rules can be used to calculate the derivative for functions of the form $f(x) = x^r$.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

TOPIC 2.6
Derivative Rules: Constant, Sum, Difference, and Constant Multiple

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.A

Calculate derivatives of familiar functions.

ESSENTIAL KNOWLEDGE

FUN-3.A.2

Sums, differences, and constant multiples of functions can be differentiated using derivative rules.

FUN-3.A.3

The power rule combined with sum, difference, and constant multiple properties can be used to find the derivatives for polynomial functions.

TOPIC 2.7
Derivatives of cos x, sin x, e^x, and ln x

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.A

Calculate derivatives of familiar functions.

ESSENTIAL KNOWLEDGE

FUN-3.A.4

Specific rules can be used to find the derivatives for sine, cosine, exponential, and logarithmic functions.

ENDURING UNDERSTANDING

LIM-3

Reasoning with definitions, theorems, and properties can be used to determine a limit.

LEARNING OBJECTIVE

LIM-3.A

Interpret a limit as a definition of a derivative.

ESSENTIAL KNOWLEDGE

LIM-3.A.1

In some cases, recognizing an expression for the definition of the derivative of a function whose derivative is known offers a strategy for determining a limit.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

TOPIC 2.8
The Product Rule

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.B

Calculate derivatives of products and quotients of differentiable functions.

ESSENTIAL KNOWLEDGE

FUN-3.B.1

Derivatives of products of differentiable functions can be found using the product rule.

TOPIC 2.9
The Quotient Rule

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.B

Calculate derivatives of products and quotients of differentiable functions.

ESSENTIAL KNOWLEDGE

FUN-3.B.2

Derivatives of quotients of differentiable functions can be found using the quotient rule.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Professional Development > **Selecting Procedures for Derivatives**
- AP Online Teacher Community Discussion > **Simplifying the Quotient Rule**

SUGGESTED SKILL

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

TOPIC 2.10
Finding the Derivatives of Tangent, Cotangent, Secant, and/or Cosecant Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.B

Calculate derivatives of products and quotients of differentiable functions.

ESSENTIAL KNOWLEDGE

FUN-3.B.3

Rearranging tangent, cotangent, secant, and cosecant functions using identities allows differentiation using derivative rules.